Exercise 9.7.4

Solve the same PDE as in Exercise 9.7.3 for a rod of length L, with position on the rod given by the variable x, with the two ends of the rod at $x = 0$ and $x = L$ kept (at all times t) at the respective temperatures $T = 1$ and $T = 0$, and with the rod initially at $T(x) = 0$, for $0 < x \leq L$.

Solution

Here we will solve the initial boundary value problem,

$$\begin{aligned}
&\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}, \quad 0 < x \leq L,\ t > 0 \\
&T(0,t) = 1 \\
&T(L,t) = 0 \\
&T(x,0) = 0.
\end{aligned}$$

Because one of the boundary conditions is inhomogeneous, the method of separation of variables cannot be applied. The ends of the rod are held at fixed temperatures, so the temperature in the rod will reach a steady state in the long run. We thus write $T(x,t)$ as the sum of a steady part and a nonsteady part: $T(x,t) = U(x) + V(x,t)$.

$$\frac{\partial}{\partial t}[U(x) + V(x,t)] = a^2 \frac{\partial^2}{\partial x^2}[U(x) + V(x,t)]$$

$$\frac{\partial V}{\partial t} = a^2 \left(\frac{d^2 U}{dx^2} + \frac{\partial^2 V}{\partial x^2} \right)$$

If we set

$$\frac{d^2 U}{dx^2} = 0$$

and assign U to satisfy the boundary conditions,

$$U(0) = 1 \quad \text{and} \quad U(L) = 0,$$

then the PDE becomes one solely for V.

$$\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2}$$

The initial and boundary conditions associated with it are as follows.

$$\begin{array}{ccccccc}
T(0,t) = 1 & \rightarrow & U(0) + V(0,t) = 1 & \rightarrow & 1 + V(0,t) = 1 & \rightarrow & V(0,t) = 0 \\
T(L,t) = 0 & \rightarrow & U(L) + V(L,t) = 0 & \rightarrow & 0 + V(L,t) = 0 & \rightarrow & V(L,t) = 0 \\
T(x,0) = 0 & \rightarrow & U(x) + V(x,0) = 0 & \rightarrow & V(x,0) = -U(x) & &
\end{array}$$

Solve the ODE for U by integrating both sides with respect to x twice.

$$U(x) = C_1 x + C_2$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned}
U(0) &= C_2 = 1 \\
U(L) &= C_1 L + C_2 = 0 \quad \rightarrow \quad C_1 = -\frac{1}{L}
\end{aligned}$$

As a result,

$$U(x) = -\frac{x}{L} + 1 = \frac{L-x}{L},$$

and the initial condition $V(x,0)$ is

$$V(x,0) = \frac{x-L}{L}.$$

Since the PDE for V and its boundary conditions are linear and homogeneous, the method of variables can be applied to solve the equation. Assume a product solution of the form $V(x,t) = X(x)\Theta(t)$ and substitute it into the PDE

$$\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)\Theta(t)] = a^2 \frac{\partial^2}{\partial x^2}[X(x)\Theta(t)]$$

and the boundary conditions.

$$\begin{array}{ccccc} V(0,t) = 0 & \rightarrow & X(0)\Theta(t) = 0 & \rightarrow & X(0) = 0 \\ V(L,t) = 0 & \rightarrow & X(L)\Theta(t) = 0 & \rightarrow & X(L) = 0 \end{array}$$

Separate variables in the PDE.

$$X\frac{d\Theta}{dt} = a^2 \Theta \frac{d^2X}{dx^2}$$

Divide both sides by $a^2X(x)\Theta(t)$. (The final answer for $V(x,t)$ will be the same regardless which side a^2 is on. Normally constants are grouped with t.)

$$\underbrace{\frac{1}{a^2\Theta}\frac{d\Theta}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t and be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{a^2\Theta}\frac{d\Theta}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation variables, the PDE for V has reduced to two ODEs—one for x and one for t.

$$\left.\begin{aligned} \frac{1}{a^2\Theta}\frac{d\Theta}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that satisfy the boundary conditions, $X(0) = 0$ and $X(L) = 0$, are called the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \alpha x + C_4 \sinh \alpha x$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_3 = 0 \\ X(L) &= C_3 \cosh \alpha L + C_4 \sinh \alpha L = 0 \end{aligned}$$

The second equation reduces to $C_4 \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_4 must be zero. The trivial solution $X(x) = 0$ is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0$$

Integrate both sides with respect to x twice.

$$X(x) = C_5 x + C_6$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_6 = 0 \\ X(L) &= C_5 L + C_6 = 0 \end{aligned}$$

The second equation reduces to $C_5 = 0$, so the trivial solution $X(x) = 0$ is obtained. Zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X(0) &= C_7 = 0 \\ X(L) &= C_7 \cos \beta L + C_8 \sin \beta L = 0 \end{aligned}$$

The second equation reduces to $C_8 \sin \beta L = 0$. In order to avoid the trivial solution, we insist that $C_8 \neq 0$. Then

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenvalues are thus $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_8 \sin \beta x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}. \end{aligned}$$

Note that n only takes on the values it does because negative values result in redundant values of λ. With this formula for λ, the ODE for T will now be solved.

$$\frac{1}{a^2\Theta} \frac{d\Theta}{dt} = -\frac{n^2\pi^2}{L^2}$$

Multiply both sides by $a^2\Theta$.

$$\frac{d\Theta}{dt} = -\frac{a^2n^2\pi^2}{L^2}\Theta$$

The general solution is written in terms of the exponential function.

$$\Theta(t) = C_9 \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right) \quad \rightarrow \quad \Theta_n(t) = \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for V is a linear combination of $X_n(x)\Theta_n(t)$ over all values of n.

$$V(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right) \sin\frac{n\pi x}{L}$$

The constants A_n are determined by applying the initial condition.

$$V(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = \frac{x-L}{L}$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{x-L}{L}\sin\frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L \frac{x-L}{L}\sin\frac{m\pi x}{L}\,dx$$

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L \frac{x-L}{L}\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left side is zero for $n \neq m$. As a result, all terms in the infinite series vanish except for the one where $n = m$.

$$A_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L \frac{x-L}{L}\sin\frac{n\pi x}{L}\,dx$$

$$A_n\left(\frac{L}{2}\right) = -\frac{L}{n\pi}$$

$$A_n = -\frac{2}{n\pi}$$

So then

$$V(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}.$$

Therefore,

$$T(x,t) = \frac{L-x}{L} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}.$$